Multidimensional perfect fluid cosmology with stable compactified internal dimensions

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Abstract. Multidimensional cosmological models in the presence of a bare cosmological constant and a perfect fluid are investigated under dimensional reduction to $D_0 = 4$ - dimensional effective models. Stable compactification of the internal spaces is achieved for a special class of perfect fluids. The external space behaves in accordance with the standard Friedmann model. Necessary restrictions on the parameters of the models are found to ensure dynamical behaviour of the external (our) universe in agreement with observations.

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1. Introduction

The large scale dynamics of the observable part of our present time universe is well described by the Friedmann model with 4-dimensional Friedmann-Robertson-Walker (FRW) metric. However, it is possible that space-time at short (Planck) distances might have a dimensionality of more than four and possess a rather complex topology [1]. String theory [2] and its recent generalizations — p-brane, M- and F-theory [3, 4] widely use this concept and give it a new foundation. From this viewpoint, it is natural to generalize the Friedmann model to multidimensional cosmological models (MCM) with topology [5]

$$M = \mathbb{R} \times M_0 \times M_1 \times \ldots \times M_n \ , \tag{1.1}$$

where for simplicity the $M_i \quad (i = 0, \ldots, n)$ can be assumed to be d_i-dimensional Einstein spaces. M_0 usually denotes the $d_0 = 3$ - dimensional external space. One of the main problems in MCM consists in the dynamical process leading from a stage with all dimensions developing on the same scale to the actual stage of the universe, where we have only four external dimensions and all internal spaces have to be compactified and contracted to sufficiently small scales, so that they are aparently unobservable. To make the internal dimensions unobservable at actual stage of the universe we have to demand their contraction to scales near to the Planck length $L_{Pl} \sim 10^{-33} cm$.

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Obviously, such a compactification should be stable. Recently [6] , we found a class of MCM possessing stable compactification of extra dimensions.

From the other hand, any realistic MCM should provide a dynamical behaviour of the external space-time in accordance with the observable universe. The phenomenological approach with a pefrect fluid as a matter source is widely used in usual 4-dimensional cosmology. According to the present day observations dynamical behaviour of the universe after inflation is well described by the standard Friedmann model [7] in the presence of a perfect fluid . Thus it might be worth-while to generalize this approach to the description of the postinflationary stage in multidimensional cosmological models. It is desirable to get models where, from one side, the internal spaces are stably compactified near Planck scales and, from the other side, the external universe behaves in accordance with the standard Friedmann model.

Here we present a toy MCM which shows a principal possibility to reach this goal. This model is out of the scope of MCM with stable compactification found in [6]. The main difference consists in an additional time-dependent term in the effective potential that provides the needed dynamical behaviour of the external space-time. This term is induced by a special type of fine-tuning of the parameters of a multicomponent perfect fluid. Although such a fine-tuning is a strong restriction on the matter content of the model, many important cases of physically interest are described by this class of perfect fluid. We note that a similar class of perfect fluids was considered in [8], where MCMs were integrated in the case of an absent cosmological constant and Ricci-flat internal spaces. As result particular solutions with static internal spaces had been obtained. According to Sec.4 of the present paper these solutions are not stable and a bare cosmological constant and internal spaces with non-vanishing curvature are necessary conditions for their stabilization. In the present paper we show that with the help of suitably chosen parameters the model can be further improved to solve two problems simultaneously. First, the internal spaces undergo stable compactification. Second, the external space behaves in accordance with the standard Friedmann model.

The paper is organized as follows. In Section 2, the general description of the considered model is given. In Section 3, the effective potential is obtained under dimensional reduction to a D_0-dimensional (usually $D_0 = 4$) effective theory in the Einstein frame. The problem of stable compactification is investigated in Section 4 for a toy model with suitably chosen parameters. Here, it is shown that the external universe behaves as the standard Friedmann model. Conclusions and references complete the paper.

2. General description of the model

We consider a multidimensional cosmological model on a manifold (1.1) in the presence of a perfect fluid and a bare cosmological constant Λ. The metric of the model is parametrized as

$$g = g_{MN} dX^M \otimes dX^N = -\exp[2\gamma(\tau)] d\tau \otimes d\tau + \sum_{i=0}^{n} \exp[2\beta^i(\tau)] g_{(i)}. \tag{2.1}$$

Manifolds M_i with the metrics $g_{(i)}$ are Einstein spaces of dimension d_i, i.e.

$$R_{mn}\left[g^{(i)}\right] = \lambda^i g^{(i)}_{mn}, \qquad m, n = 1, \ldots, d_i \tag{2.2}$$

and

$$R\left[g^{(i)}\right] = \lambda^i d_i \equiv R_i. \tag{2.3}$$

In the case of constant curvature spaces parameters λ^i are normalized as $\lambda^i = k_i(d_i - 1)$ with $k_i = \pm 1, 0$. The scalar curvature corresponding to the metric (2.1) reads

$$R = \sum_{i=0}^{n} R_i \exp\left(-2\beta^i\right) + \exp\left(-2\gamma\right) \sum_{i=0}^{n} d_i \left[2\ddot{\beta}^i - 2\dot{\gamma}\dot{\beta}^i + \left(\dot{\beta}^i\right)^2 + \dot{\beta}^i \sum_{j=0}^{n} d_j \dot{\beta}^j\right]. \quad (2.4)$$

Matter fields we take into account in a phenomenological way as a $m-$component perfect fluid with energy-momentum tensor

$$T_N^M = \sum_{a=1}^{m} T^{(a)}{}_N^M, \quad (2.5)$$

$$T^{(a)}{}_N^M = \mathrm{diag}\left(-\rho^{(a)}(\tau), \underbrace{P_0^{(a)}(\tau), \ldots, P_0^{(a)}(\tau)}_{d_0 \text{ times}}, \ldots, \underbrace{P_n^{(a)}(\tau), \ldots, P_n^{(a)}(\tau)}_{d_n \text{ times}}\right) \quad (2.6)$$

and equations of state

$$P_i^{(a)} = \left(\alpha_i^{(a)} - 1\right) \rho^{(a)}, \qquad i = 0, \ldots, n, \quad a = 1, \ldots, m. \quad (2.7)$$

It is easy to see that physical values of $\alpha_i^{(a)}$ according to $-\rho^{(a)} \leq P_i^{(a)} \leq \rho^{(a)}$ run the region $0 \leq \alpha_i^{(a)} \leq 2$. The conservation equations we impose on each component separately

$$T^{(a)}{}_{N;M}^M = 0. \quad (2.8)$$

Denoting by an overdot differentiation with respect to time τ, these equations read for the tensors (2.6)

$$\dot{\rho}^{(a)} + \sum_{i=0}^{n} d_i \dot{\beta}^i \left(\rho^{(a)} + P_i^{(a)}\right) = 0 \quad (2.9)$$

and have according to (2.7) the simple integrals

$$\rho^{(a)}(\tau) = A^{(a)} \prod_{i=0}^{n} a_i^{-d_i \alpha_i^{(a)}}, \quad (2.10)$$

where $a_i \equiv e^{\beta^i}$ are scale factors of M_i and $A^{(a)}$ are constants of integration. It is not difficult to verify that the Einstein equations with the energy-momentum tensor (2.5)-(2.10) are equivalent to the Euler-Lagrange equations for the Lagrangian [9],[10]

$$L = \frac{1}{2} e^{-\gamma + \gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j - e^{\gamma + \gamma_0} \left(-\frac{1}{2} \sum_{i=0}^{n} R_i e^{-2\beta^i} + \kappa^2 \sum_{a=1}^{m} \rho^{(a)} + \Lambda\right). \quad (2.11)$$

Here we use the notation $\gamma_0 = \sum_0^n d_i \beta^i$, Λ is a cosmological constant and κ^2 is a $D = \sum_0^n d_i + 1$ - dimensional gravitational constant. The components of the minisuperspace metric read [5]

$$G_{ij} = d_i \delta_{ij} - d_i d_j \ . \quad (2.12)$$

The Lagrangian (2.11) can be obtained by dimensional reduction of the action

$$S = \frac{1}{2\kappa^2} \int\limits_M d^D x \sqrt{|g|} \left\{R[g] - 2\Lambda\right\} - \int\limits_M d^D x \sqrt{|g|} \rho + S_{YGH} = \frac{\mu}{\kappa^2} \int d\tau L. \quad (2.13)$$

S_{YGH} is the standard York-Gibbons-Hawking boundary term and $\mu = \prod_{i=0}^n V_i$, where V_i is the volume of M_i (with unit scale factors) : $V_i = \mathrm{vol}\,(M_i) = \int\limits_{M_i} d^{d_i} y \sqrt{|g^{(i)}|}$.

3. The effective potential

Let us slightly generalize this model to the inhomogeneous case supposing that the scale factors $\beta^i = \beta^i(x)$ $(i = 0, \ldots, n)$ are functions of the coordinates x, where x are defined on the $D_0 = (1+d_0)$ - dimensional external space-time manifold $\bar{M}_0 = \mathbb{R} \times M_0$ with the metric

$$\bar{g}^{(0)} = \bar{g}^{(0)}_{\mu\nu} dx^\mu \otimes dx^\nu = -e^{2\gamma} d\tau^2 + e^{2\beta^0(x)} g^{(0)} \ . \qquad (3.1)$$

After conformal transformation of the external space-time metric from the Brans-Dicke to the Einstein frame:

$$\begin{aligned} g = g_{MN} dX^M \otimes dX^N &= \bar{g}^{(0)} + \sum_{i=1}^{n} \exp\left[2\beta^i(x)\right] g^{(i)} \\ &= \Omega^2 \tilde{g}^{(0)} + \sum_{i=1}^{n} \exp\left[2\beta^i(x)\right] g^{(i)}, \end{aligned} \qquad (3.2)$$

where

$$\Omega^2 = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{-\frac{2}{D_0-2}} , \qquad (3.3)$$

the dimensionally reduced action (2.13) reads

$$S = \frac{1}{2\kappa_0^2} \int\limits_{\bar{M}_0} d^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] - \bar{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_\mu \beta^i \, \partial_\nu \beta^j - 2U_{eff} \right\} , \qquad (3.4)$$

where $\kappa_0^2 = \kappa^2 / V_I$ is the D_0-dimensional gravitational constant, $V_I = \prod_{i=1}^{n} V_i$, $\bar{G}_{ij}$ is the midisuperspace metric with the components

$$\bar{G}_{ij} = d_i \delta_{ij} + \frac{1}{D_0 - 2} d_i d_j \ , \quad i, j = 1, \ldots, n \qquad (3.5)$$

and the effective potential U_{eff} reads

$$U_{eff} = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2\beta^i} + \Lambda + \kappa^2 \sum_{a=1}^{m} \rho^{(a)} \right] . \qquad (3.6)$$

The effective action (3.4) has the form of a usual 4-dimensional (if $d_0 = 3$) theory and describes a self-gravitating $\sigma-$model with self-interaction. The internal scale factors play the role of scalar fields (dilatons in the starting Brans-Dicke frame) satisfying the wave equation

$$\bar{G}_{ij} \Box \beta^j \equiv \frac{1}{\sqrt{|\tilde{g}^{(0)}|}} \partial_\mu \left(\sqrt{|\tilde{g}^{(0)}|} \bar{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_\nu \beta^j \right) = \frac{\partial U_{eff}}{\partial \beta^i} \ . \qquad (3.7)$$

In the Einstein frame the theory assumes the most natural form [11], [12] and beginning from this point the external space-time metric $\tilde{g}^{(0)}$ is considered as the physical one. For this metric we adopt following ansatz:

$$\tilde{g}^{(0)} = \Omega^{-2} \bar{g}^{(0)} = \tilde{g}^{(0)}_{\mu\nu} dx^\mu \otimes dx^\nu = -e^{2\tilde{\gamma}} d\tilde{\tau} \otimes d\tilde{\tau} + e^{2\tilde{\beta}^0(x)} g^{(0)} \ . \qquad (3.8)$$

Thus external scale factors in the Brans-Dicke frame $a_0 = e^{\beta^0} \equiv a$ and in the Einstein frame $\tilde{a}_0 = e^{\tilde{\beta}^0} \equiv \tilde{a}$ are connected with each other by the relation

$$a = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{-\frac{1}{D_0-2}} \tilde{a}. \qquad (3.9)$$

The energy densities $\rho^{(a)}$ of the perfect fluid components are given by (2.10) and with the help of relation (3.9) can be rewritten as

$$\rho^{(a)} = \rho_0^{(a)} \prod_{i=1}^{n} a_i^{-\xi_i^{(a)}} , \tag{3.10}$$

where

$$\rho_0^{(a)} = A^{(a)} \frac{1}{\tilde{a}^{\alpha_0^{(a)} d_0}} \tag{3.11}$$

and

$$\xi_i^{(a)} = d_i \left(\alpha_i^{(a)} - \frac{\alpha_0^{(a)} d_0}{d_0 - 1} \right) . \tag{3.12}$$

In the case of one internal space ($n = 1$) the action and the effective potential are respectively

$$S = \frac{1}{2\kappa_0^2} \int\limits_{\tilde{M}_0} d^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] - \tilde{g}^{(0)\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 2U_{eff} \right\} \tag{3.13}$$

and

$$U_{eff} = e^{2\varphi \left[\frac{d_1}{(D-2)(D_0-2)} \right]^{1/2}} \left[-\frac{1}{2} R_1 e^{2\varphi \left[\frac{D_0-2}{d_1(D-2)} \right]^{1/2}} + \Lambda + \kappa^2 \rho(\tilde{a}, \varphi) \right] , \tag{3.14}$$

where we redefined the dilaton field as

$$\varphi \equiv - \sqrt{\frac{d_1(D-2)}{D_0 - 2}} \beta^1 . \tag{3.15}$$

Let us split the scalar fields $\beta^i(x)$ in equationss (3.4) and (3.6) in a background component $\bar{\beta}^i(x)$ and a small perturbational (fluctuation) component $\eta^i(x)$

$$\beta^i(x) = \bar{\beta}^i(x) + \eta^i(x). \tag{3.16}$$

Assuming that such a splitting procedure is well defined we get the corresponding equations of motion from (3.7) as

$$\Box \bar{\beta}^i = \left[\bar{G}^{-1}\right]^{ij} b_j(\bar{\beta}) \tag{3.17}$$

and

$$\Box \eta^i = \left[\bar{G}^{-1}\right]^{ij} a_{jk}(\bar{\beta}) \eta^k , \tag{3.18}$$

where

$$a_{ij} := \frac{\partial^2 U_{eff}}{\partial \beta^i \partial \beta^j}, \quad b_i := \frac{\partial U_{eff}}{\partial \beta^i} . \tag{3.19}$$

With the help of an appropriate background depending $SO(n)-$rotation $S = S(\bar{\beta})$ we can diagonalize matrix $\left[\bar{G}^{-1}A\right]^i_k \equiv \left[\bar{G}^{-1}\right]^{ij} a_{jk}(\bar{\beta})$ and rewrite (3.18) in terms of normal modes $\psi = S^{-1}\eta$:

$$\tilde{g}^{(0)\mu\nu} D_\mu D_\nu \psi = S^{-1} \bar{G}^{-1} A S \psi \stackrel{def}{=} M^2 \psi, \tag{3.20}$$

where M^2 is a background depending diagonal mass matrix

$$M^2 = diag\left[m_1^2(\bar{\beta}), \ldots, m_n^2(\bar{\beta})\right] . \tag{3.21}$$

D_μ denotes a covariant derivative

$$D_\mu := \partial_\mu + \Gamma_\mu + A_\mu \ , \quad A_\mu := S^{-1}\partial_\mu S \tag{3.22}$$

with $\Gamma_\mu + A_\mu$ as connection on the fibre bundle $E(\bar{M}_0, \mathbb{R}^{D_0} \oplus \mathbb{R}^n)$ consisting of the base manifold $\bar{M}_0$ and vector spaces $\mathbb{R}^{D_0}_x \oplus \mathbb{R}^n_x = T_x\bar{M}_0 \oplus \left\{\left(\eta^1(x), \ldots, \eta^n(x)\right)\right\}$ as fibres. So, the background components $\bar{\beta}^i(x)$ via the effective potential U_{eff} and its Hessian a_{ij} play the role of a medium for the fluctuational components $\psi^i(x)$. Propagating in $\bar{M}_0$ filled with this medium the excitational modes (gravitational excitons [6]) change their masses as well as the direction of their "polarization" defined by the unit vector in the fibre space

$$\xi(x) := \frac{\psi(x)}{|\psi(x)|} \in S^{n-1} \subset \mathbb{R}^n, \tag{3.23}$$

where S^{n-1} denotes the $(n-1)-$dimensional sphere. For considerations on interactions of gravitational excitons with gauge fields and corresponding observable effects we refer to [13].

We note that in the general case, when $m_i^2(\bar{\beta}) \neq m_j^2(\bar{\beta})$, $\quad i \neq j$, due to the lack of $SO(n)-$invariance of (3.20) the connection A_μ itself cannot be interpreted as a $SO(n)-$gauge connection in pure gauge. This is only possible for $M^2 = m^2_{exci} I_n$, with I_n the unit matrix. Then a transformation

$$\begin{array}{rrcl} U: & \psi & \mapsto & \tilde{\psi} = U\psi \\ & A_\mu & \mapsto & \tilde{A}_\mu = UA_\mu U^{-1} - (\partial_\mu U)U^{-1} \\ & D_\mu & \mapsto & \tilde{D}_\mu = \partial_\mu + \Gamma_\mu + \tilde{A}_\mu \\ & D_\mu\psi & \mapsto & \tilde{D}_\mu\tilde{\psi} = UD_\mu\psi \end{array} \tag{3.24}$$

leaves (3.20) invariant due to $M^2 \mapsto \tilde{M}^2 = UM^2U^{-1} = M^2$, and U is indeed a gauge transformation.

Further from (3.20) it is clear that a consideration of the excitational modes makes only sense if the characteristic space-time scales $L_{\bar{\beta}}$ and L_ψ of the variations of the background fields $\bar{\beta}^i$ and the excitons ψ^i are of different order: $L_{\bar{\beta}} \gg L_\psi$. (Otherwise non-perturbative techniques should be applied.) Coverring the external space-time with domains Ω_c of intermedium characteristic length $L_c \approx |\Omega_c|^{1/(d_0+1)}$, $L_{\bar{\beta}} \gg L_c \gg L_\psi$ we can in a crude approximation replace the background fields $\bar{\beta}^i(x)$ in Ω_c by constants $\bar{\beta}^i_c$. According to (3.17), (3.20) and due to the regularity of the minisuperspace metric $\bar{G}_{ij}$ this implies an extremum condition on the effective potential in Ω_c

$$\left.\frac{\partial U_{eff}}{\partial \beta^i}\right|_{\vec{\beta}_c} = 0, \tag{3.25}$$

as well as a vanishing connection $A_\mu = 0$ and the constancy of matrix M^2. The only extremum that provides the constancy of $\bar{\beta}^i_c$ under perturbations ψ^i is a minimum and the exciton masses must be non-negative $m^2_{(c)i} := m_i^2(\vec{\beta}_c) \geq 0$ with at least one of them strictly positive. (The case of $m^2_{(c)i} = 0,\ m^2_{(c)j} > 0$ for some i, j corresponds to degenerate minima, as e.g. given in Sombrero-like potentials. The massless modes are similar to Goldstone bosons.)

Models with constant background fields on $\Omega_c = \bar{M}_0$ and with effective potentials U_{eff} depending only on the internal scale factors have been considered in [6, 14]. The corresponding action functional reads in this case:

$$S = \frac{1}{2\kappa_0^2} \int\limits_{\tilde{M}_0} d^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] - 2\Lambda_{(c)eff} \right\} +$$

$$+ \sum_{i=1}^{n} \frac{1}{2} \int\limits_{\tilde{M}_0} d^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ -\tilde{g}^{(0)\mu\nu} \psi^i_{,\mu} \psi^i_{,\nu} - m^2_{(c)i} \psi^i \psi^i \right\} \ , \qquad (3.26)$$

where the effective cosmological constant $\Lambda_{(c)eff}$ is connected with the stable compactification position $a_{(c)i} = \exp \bar{\beta}^i_c$ by the relation $\Lambda_{(c)eff} \equiv U_{eff}(\vec{\beta}_c)$. From a physical point of view it is clear that the effective potential should satisfy following conditions:

$$\begin{aligned} &(i) \qquad a_{(c)i} \gtrsim \ L_{Pl} \ , \\ &(ii) \qquad m_{(c)i} \ \leq \ M_{Pl} \ , \\ &(iii) \quad \Lambda_{(c)eff} \to 0 \ . \end{aligned}$$

The first condition expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. The second condition means that the curvature of the effective potential should be less than Planckian one. Of course, gravitational excitons can be excited at the present time if $m_i \ll M_{Pl}$. The third condition reflects the fact that the cosmological constant at the present time is very small: $|\Lambda| \leq 10^{-56}\mathrm{cm}^{-2} \approx 10^{-121}\Lambda_{Pl}$ where $\Lambda_{Pl} = L_{Pl}^{-2}$. Strictly speaking, in the case that the potential has several minima ($c > 1$) we can demand $a_{(c)i} \sim L_{Pl}$ and $\Lambda_{(c)eff} \to 0$ only for one of the minima to which corresponds the present state of the universe. For all other minima it may be $a_{(c)i} \gg L_{Pl}$ and $|\Lambda_{(c)eff}| \gg 0$.

4. The model

A general analysis of the internal spaces stable compactification for MCM with the perfect fluid (2.10) is carried out in our paper [15]. In the present paper we investigate a particular class of effective potentials (3.6) with separating scale factor contributions from internal and external factor spaces

$$U_{eff} = \underbrace{\left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2\beta^i} + \Lambda \right]}_{U_{int}} + \underbrace{\kappa^2 \sum_{a=1}^{m} \rho_0^{(a)}}_{U_{ext}} . \quad (4.1)$$

We will show below, that such a separation on the one hand provides a stable compactification of the internal factor spaces due to a minimum of the first term $U_{int} = U_{int}(\beta^1, \ldots, \beta^n)$ as well as a dynamical behaviour of the external factor space due to $U_{ext} = U_{ext}(\tilde{\beta}^0)$. On the other hand this separation crucially simplifies the calculations and allows an exact analysis. The price that we have to pay for the separation is a fine-tuning of the parameters of the multicomponent perfect fluid

$$\begin{aligned} \alpha_0^{(a)} &= \tfrac{2}{d_0} + \tfrac{d_0 - 1}{d_0} \alpha^{(a)} \\ \alpha_i^{(a)} &= \alpha^{(a)}, \quad i = 1, \ldots, n, \quad a = 1, \ldots, m. \end{aligned} \qquad (4.2)$$

Only in this case we have

$$\xi_i^{(a)} = -\frac{2d_i}{d_0 - 1} \tag{4.3}$$

yielding the compensation of the exponential prefactor for the perfect fluid term in the effective potential (3.6). The corresponding components $\rho_0^{(a)}$ read, respectively,

$$\rho_0^{(a)} = A^{(a)} \frac{1}{\tilde{a}^{2+(d_0-1)\alpha^{(a)}}} \ . \tag{4.4}$$

Although the fine-tuning (4.2) is a strong restriction, there exist some important particular models that belong to this class of multicomponent perfect fluids. For example, if $\alpha^{(a)} = 1$ the a-th component of the perfect fluid describes radiation in the space M_0 and dust in the spaces $M_1, \ldots, M_n$. This kind of perfect fluid satisfies the condition $\sum_{i=0}^n d_i \alpha_i^{(a)} = D$ and is called superradiation [16]. If $\alpha^{(a)} = 2$ we obtain the ultra-stiff matter in all $M_i\,(i = 0 \ldots, n)$ which is equivalent, e.g., to a massless minimally coupled free scalar field. In the case $\alpha^{(a)} = 0$ we get the equation of state $P_0^{(a)} = \left[(2-d_0)/d_0\right]\rho^{(a)}$ in the external space M_0 which describes a gas of cosmic strings if $d_0 = 3 :\ P^{(a)} = -\frac{1}{3}\rho^{(a)}$ [17] and vacuum in the internal spaces $M_1, \ldots, M_n$. If $\alpha^{(a)} = 1/2$ and $d_0 = 3$ we obtain dust in the external space M_0 and a matter with equation of state $P_i^{(a)} = -\frac{1}{2}\rho^{(a)}$ in the internal spaces $M_i, \quad i = 1, \ldots, n$.

Let us first consider the conditions for the existence of a minimum of the potential $U_{int}(\beta^1, \ldots, \beta^n)$. According to reference [14] potentials U_{int} of type (4.1) have a single minimum if the bare cosmological constant and the curvature scalars of the internal spaces are negative $R_i, \Lambda < 0$. The scale factors $\left\{\beta_c^i\right\}_{i=1}^n$ at the minimum position of the effective potential are connected by a fine-tuning condition

$$\frac{R_i}{d_i} e^{-2\beta_c^i} = \frac{2\Lambda}{D-2} \equiv \widetilde{C}, \qquad i = 1, \ldots, n \tag{4.5}$$

and the masses squared of the corresponding gravitational excitons are degenerate and given as

$$\begin{aligned} m_1^2 = \ldots = m_n^2 = m_{exci}^2 \quad &= -\tfrac{4\Lambda}{D-2} \exp\left[-\tfrac{2}{d_0-1}\textstyle\sum_{i=1}^n d_i \beta_c^i\right] \\ &= 2\left|\widetilde{C}\right|^{\frac{D-2}{d_0-1}} \textstyle\prod_{i=1}^n \left|\tfrac{d_i}{R_i}\right|^{\frac{d_i}{d_0-1}} \ . \end{aligned} \tag{4.6}$$

Further it was shown in reference [14] that the value of the potential U_{int} at the minimum is connected with the exciton mass by the relation

$$\Lambda_{int} := U_{int}(\beta_c^1, \ldots, \beta_c^n) = -\frac{d_0 - 1}{4} m_{exci}^2 \ . \tag{4.7}$$

From equations (4.5), (4.6) we see that exciton masses and minimum position $a_{(c)i} = \exp \bar{\beta}_c^i$ are constants that solely depend on the value of the bare cosmological constant Λ, the (constant) curvature scalars R_i and dimensions d_i of the internal factor spaces. This means that we have automatically $\Omega_c = \bar{M}_0$ from the very onset of the model. Hence the exciton approach in the present linear form breaks down only when the excitations become too strong so that higher order terms must be included in the consideration or the phenomenological perfect fluid approximation itself becomes inapplicable.

Let us now turn to the dynamical behaviour of the external factor space. For simplicity we consider the zero order approximation, when all excitations are freezed, in the homogeneous case: $\tilde{\gamma} = \tilde{\gamma}(\tilde{\tau})$ and $\tilde{\beta} = \tilde{\beta}(\tilde{\tau})$. Then the action functional (3.26) with

$$U_{(c)eff} \equiv U_{eff}\left[\vec{\beta}_c, \tilde{\beta}(\tilde{\tau})\right] = U_{int}(\beta_c^1, \ldots, \beta_c^n) + U_{ext}\left[\tilde{\beta}(\tilde{\tau})\right] \equiv \Lambda_{int} + \bar{\rho}_0(\tilde{\tau}) \tag{4.8}$$

after dimensional reduction reads:

$$\begin{aligned} S = {} & \frac{1}{2\kappa_0^2}\int\limits_{\tilde{M}_0} d^{D_0}x\sqrt{|\tilde{g}^{(0)}|}\left\{\tilde{R}\left[\tilde{g}^{(0)}\right] - 2U_{(c)eff}\right\} = \\ & = \frac{V_0}{2\kappa_0^2}\int d\tilde{\tau}\left\{e^{\tilde{\gamma}+d_0\tilde{\beta}}e^{-2\tilde{\beta}}R[g^{(0)}] + d_0(1-d_0)e^{-\tilde{\gamma}+d_0\tilde{\beta}}\left(\frac{d\tilde{\beta}}{d\tilde{\tau}}\right)^2\right. \\ & \left. -2e^{\tilde{\gamma}+d_0\tilde{\beta}}\left(\Lambda_{int} + \bar{\rho}_0\right)\right\} + \frac{V_0}{2\kappa_0^2}d_0\int d\tilde{\tau}\frac{d}{d\tilde{\tau}}\left(e^{-\tilde{\gamma}+d_0\tilde{\beta}}\frac{d\tilde{\beta}}{d\tilde{\tau}}\right), \end{aligned} \tag{4.9}$$

where usually $R[g^{(0)}] = kd_0(d_0-1), \quad k = \pm 1, 0$. The constraint equation $\partial L/\partial\tilde{\gamma} = 0$ in the synchronous time gauge $\tilde{\gamma} = 0$ yields

$$\left(\frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = -\frac{k}{\tilde{a}^2} + \frac{2}{d_0(d_0-1)}\left(\Lambda_{int} + \bar{\rho}_0(\tilde{a})\right), \tag{4.10}$$

which results in

$$\begin{aligned} \tilde{t} + const = {} & \int \frac{d\tilde{a}}{\left[-k + \frac{2\Lambda_{int}}{d_0(d_0-1)}\tilde{a}^2 + \frac{2\kappa^2}{d_0(d_0-1)}\sum_{a=1}^m \frac{A^{(a)}}{\tilde{a}^{(d_0-1)\alpha^{(a)}}}\right]^{1/2}}, \\ = {} & \int \frac{d\tilde{a}}{\left[-k + \frac{\Lambda_{int}}{3}\tilde{a}^2 + \frac{\kappa^2}{3}\sum_{a=1}^m \frac{A^{(a)}}{\tilde{a}^{2\alpha^{(a)}}}\right]^{1/2}}, \end{aligned} \tag{4.11}$$

where in the last line we put $d_0 = 3$.

Thus in the zero order approximation we arrived at a Friedmann model in the presence of negative cosmological constant Λ_{int} and a multicomponent perfect fluid. The perfect fluid has the form of a gas of cosmic strings for $\alpha^{(a)} = 0$, dust for $\alpha^{(a)} = 1/2$ and radiation for $\alpha^{(a)} = 1$. As $0 \le \alpha^{(a)} \le 2$, the cosmological constant plays a role only for large $\tilde{a}$ and because of the negative sign of Λ_{int} the universe has a turning point at the maximum of $\tilde{a}$. To be consistent with present time observation we should take

$$|\Lambda_{int}| \le 10^{-121}\Lambda_{Pl}. \tag{4.12}$$

We note that due to (4.11) and in contrast with (3.26) the minimum value $U_{(c)eff}$ of the effective potential in (4.8) cannot be interpreted as a cosmological constant, even as a time dependent one. Coming back to the gravitational excitons we see that according to (4.7) the upper bound (4.12) on the effective cosmological constant leads to ultra-light particles with mass $m_{exci} \le 10^{-60}M_{Pl} \sim 10^{-32}eV$. This is much less than the cosmic background radiation temperature at the present time $T_0 \sim 10^{-4}eV$. It is clear that such light particles up to present time behave as radiation and can be taken into account as an additional term $\rho_r = \frac{\kappa_0^2 A_r/3}{\tilde{a}^2}$ in (4.11). It can be easily seen that we reconstruct the standard scenario if we consider the one-component ($m = 1$) case with $\alpha^{(1)} = 1/2$, $\kappa^2 A^{(1)} \sim 10^{61}$ and $\kappa_0^2 A_r \sim 10^{117}$. Here we have at early stages a radiation dominated universe and a dust dominated universe at later stages of its evolution.

For completeness we note that via equations (4.6) and (4.7) the value of the effective cosmological constant has a crucial influence on the relation between the compactification scales of the internal factor spaces and their dimensions. In the case of only one internal negative curvature space $M_1 = H^{d_1}/\Gamma$ with $R_1 = -d_1(d_1-1)$ and compactification scale $a_{(c)1} = 10L_{Pl}$ we have e.g. the relation [14] $\Lambda_{int} = -(d_1-1)10^{-2(d_1+2)}L_{Pl}$, so that the bound (4.12) implies a dimension of this space of at least $d_1 = 59$. Taking instead of one internal space a set of 2-dimensional hyperbolic $g-$tori $\left\{M_i = H^2/\Gamma\right\}_{i=1}^{n}$ [18] with compactification scale $a_{(c)i} = 10^2 L_{Pl}$ it is easy to check [14] that we need at least $n = 29$ such spaces to satisfy (4.12).

Of course, other values of the cosmological constant lead to other exciton masses and compactification - dimensionality relations. So, it is also possible to get models with much more heavier gravitational excitons. For $\Lambda_{int} = -10^{-8}\Lambda_{Pl}$ we have e.g. $m = 10^{-4}M_{Pl}$ and the excitons are very heavy particles that should be considered as a cold dark matter. If we take the one-component case $\alpha^{(1)} = 1$ we get at early times a radiation dominated universe with smooth transition to a cold dark matter dominated universe at later stages. But for this example it is necessary to introduce a mechanism that provides a reduction of the huge cosmological constant to the observable value $10^{-121}\Lambda_{Pl}$.

5. Conclusion

In the present paper we considered multidimensional cosmological models (MCM) with a bare cosmological constant and a perfect fluid as a matter source. It can be easily seen that there are only two classes of perfect fluids with stably compactified internal spaces. These kind of solutions are of utmost interest because an absent time variation of the fundamental constants in experiments [19, 20] shows that at the present time the extra dimensions, if they exist, should be static or nearly static.

The first class (see [6], [14]) consists of models with $\alpha_0^{(a)} = 0$. It leads to the vacuum equation of state in the external space M_0. All other $\alpha_i^{(a)} (i = 1, \dots, n)$ can take arbitrary values. This model can be used for a phemenological description of a muitidimensional inflationary universe with smooth transition to a matter dominated stage.

In the present paper we found a second class of perfect fluid models with stable internal spaces. For these models the stability is induced by a fine-tuning of the equation of state of the perfect fluid in the external and internal spaces (4.2). This class includes many important particular models and allows considerations of perfect fluids with different equations of state in the external space, among them also such that result in a Friedmann-like dynamics. Thus, this class of models can be applied for the description of the postinflationary stage in multidimensional cosmology. For the considered models we found necessary restrictions on the parameters which, from the one hand, ensure stable compactification of the internal spaces near Planck length and, from the other hand, guarantee dynamical behaviour of the external (our) universe in accordance with the standard scenario for the Friedmann model.

This toy model gives a promising example of a multidimensional cosmological model which is not in contradiction to observations.

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References

[1] Wheeler J A 1962 *Geometrodynamics* (New York: Academic).
[2] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory* (Cambridge: Cambridge Univ. Press).
[3] Strominger A and Vafa C 1996 *Phys.Lett.* **B379** 99, (hep-th/9601029).
[4] Duff M J 1996 *Int.J.Mod.Phys.* **A11** 5623, (hep-th/9608117).
[5] Ivashchuk V D, Melnikov V N and Zhuk A I 1989 *Nuovo Cimento* **B104** 575.
[6] Günther U and Zhuk A 1997 *Phys. Rev.* **D56** 6391 - 6402, (gr-qc/9706050).
[7] Friedmann A 1924 *Z. Phys.* **10** 377, **21** 326.
[8] Kasper U and Zhuk A 1996 *Gen.Rel.Grav.* **28** 1269 - 1292.
[9] Ivashchuk V D and Melnikov V N 1995 *Class. Quant. Grav.* **12** 809.
[10] Zhuk A 1996 *Class. Quant. Grav.* **13** 2163 - 2178.
[11] Cho Y M 1992 *Phys. Rev. Lett.* **68** 3133 - 3136.
[12] Litterio M, Sokolowski L M, Golda Z A, Amendola L and Dyrek A 1996 *Phys. Lett.* **B382** 45 - 52.
[13] Günther U and Zhuk A *Gauge fields and gravitational excitons from extra dimensions*, (in preparation).
[14] Günther U and Zhuk A, *Stable compactification and gravitational excitons from extra dimensions* (Proc. Workshop "Modern Modified Theories of Gravitation and Cosmology", Beer Sheva, Israel, June 29 - 30, 1997), (gr-qc/9710086).
[15] Günther U, Lishchuk S and Zhuk A *On the possibility of quasi-stable compactification of extra dimensions*, (in preparation).
[16] Liebscher D-E and Bleyer U 1985 *Gen.Rel.Grav.* **17** 989.
[17] Spergel D and Pen U-L 1997 *Astrophys. J.* **491** L67 - L71, (astro-ph/9611198).
[18] Lachieze-Rey M and Luminet J-P 1995 *Phys.Rep.* **254** 135 -214.
[19] Marciano W J 1984 *Phys.Rev.Lett.* **52** 489 - 491.
[20] Kolb E W, Perry M J and Walker T P 1986 *Phys.Rev.* **D33** 869- 871.